Differentiation Rules

Limit Definition of a Derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

Properties of Derivatives

Let c be a constant.

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}[cf(x)] = cf'(x)$$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

Differentiation Rules

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = -\frac{f'(x)}{[f(x)]^2}, \quad f(x) \neq 0$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\, g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{x^2+1} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{x^2+1}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Derivatives of Exponential and Logarithmic Functions

Let b be a positive constant with $b \neq 1$.

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(b^x) = b^x \ln b$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \qquad \frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

Implicit Differentiation

To differentiate an implicit equation, use the following steps:

1. Differentiate both sides of the implicit equation with respect to x.
2. On the left side, collect all the terms containing the factor dy/dx. Move all the other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

Differentiating Inverse Functions

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \frac{dy}{dx} = \frac{1}{dx/dy}$$

Linearization and Differentials

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$dy = f'(x)\,dx$$

Logarithmic Differentiation

To use Logarithmic Differentiation to find $\frac{dy}{dx}$ for $y = f(x)$, use the following steps:

1. Take the natural logarithm of both sides to get
$$\ln y = \ln[f(x)]$$
Use logarithm laws to expand $\ln[f(x)]$ into a sum or difference of logarithms.
2. Differentiate both sides of $\ln y = \ln[f(x)]$ implicitly; the left side becomes $\frac{1}{y}\frac{dy}{dx}$.
3. Solve for $\frac{dy}{dx}$ by multiplying both sides by $y = f(x)$.

Exponential Function as a Limit

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Definitions of Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

Inverse Hyperbolic Functions

Identity	Domain
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1, 1)$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}}$$